Advection of a passive scalar near two dimensions

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The model of advection of a passive scalar in turbulent velocity field generated by the stochastically forced Navier-Stokes equation has been studied by means of the quantum field theoretical renormalization group near two dimensions. A perturbative two-parameter expansion scheme, the parameters of which are the deviation of the spatial dimension and the deviation of the exponent of the powerlike correlation function of the random force from their critical values, has been used in one-loop approximation. It is shown that the fixed points of the renormalization group are independent of the parameters of random injection of the passive scalar. The asymptotic behavior of velocity-velocity and scalar-scalar correlation functions has been calculated at leading order in the two-parameter expansion at the kinetic fixed point associated with the Kolmogorov scaling regime. In this regime the values of the inverse Prandtl number, the Kolmogorov constant, and the Obukhov-Corrsin constant have been calculated at leading order in the double expansion for $d\geqslant 2$. [S1063-651X(99)07604-7]

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I. INTRODUCTION

Systems with a large number of degrees of freedom display similar behavior in certain asymptotic regimes independently of numerous microscopic details of the system. In the theory of strongly developed turbulence this *universality* is connected with long-distance asymptotics of velocity correlation functions. The main indication of the universality in the turbulence comes from the celebrated Kolmogorov scaling theory [1], which describes the large-scale behavior of velocity structure functions.

Recently, the problem of universality has been actively studied in the Kraichnan model [2–9], which is the simplest model of turbulent advection of a passive scalar. In the Kraichnan model the random velocity field is assumed to be decorrelated in time, which has led to significant progress in the closed-form solution for the equal-time correlation functions of the passive scalar. This assumption is, however, far from the real behavior of the velocity field in the developed turbulence.

In this paper we study the problem of the advection of the passive scalar using a random velocity field generated by the stochastically forced Navier-Stokes equation [10], which has been widely used to produce a stochastic velocity field with the Kolmogorov scaling behavior obtained by the use of the field-theoretic renormalization group [11,12]. The passive scalar problem has already been treated within the renormalization-group (RG) approach of the randomly forced Navier-Stokes equation for both the local [13] and long-range [14] correlations of the random force, but without random pumping of the passive scalar, due to which the behavior of the correlation functions of the passive scalar was not addressed at all.

The RG method represents a powerful, general, and systematic technique in the study of asymptotic properties in the quantum field theory, from which it has been transferred to the study of static critical phenomena and also critical dynamics [15,16]. Since the late 1970s several authors [11–13,17,18] have applied the RG method to the model of randomly stirred fluid. This model, which describes hydrodynamic systems randomly stirred up by forces active at large spatial scales, is considered to be close to the statistical behavior of turbulent velocity modes at very high Reynolds numbers [19,20]. Here the application of the RG method allows an investigation with minimal empirical input and adjustable parameters.

This paper is organized as follows: Section II starts from the functional formulation of the generation of turbulent velocity field by the stochastic Navier-Stokes equation followed by the description of the passive scalar problem in this setup. This formulation is convenient for the analysis based on the RG approach, the details of which are described in Sec. III. The Kolmogorov and Obukhov-Corrsin constants are calculated in Sec. IV. In Sec. V the conclusions are presented.

II. FUNCTIONAL FORMULATION OF THE PASSIVE SCALAR PROBLEM

We recall the basic features of the generation of the random velocity field by the use of the randomly forced Navier-Stokes equation.

The stochastic evolution of the local velocity field $\mathbf{v}=\mathbf{v}(x)$, $x=(\mathbf{x},t)$ of an incompressible fluid can be described by the transverse part of the Navier-Stokes equation,

$$\mathcal{N}(\{\mathbf{v}\};\nu_0)\equiv\partial_t\mathbf{v}+P(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}-\nu_0\nabla^2\mathbf{v}=\mathbf{f}^v, \tag{2.1}$$

and the incompressibility conditions,

$$\boldsymbol{\nabla}\cdot\mathbf{v}=0,\quad \boldsymbol{\nabla}\cdot\mathbf{f}^v=0. \tag{2.2}$$

In Eq. (2.1) P is the transverse projection operator and ν_0 is the kinematic viscosity. The subscript 0 refers to unrenormalized quantities of the model, which are the physical ones in the statistical applications of the renormalization group. We denote the corresponding renormalized parameters by the same symbols without a subscript. The statistics of $\mathbf{v}$ is completely determined by the nonlinear equations (2.1) and (2.2) and certain assumptions about the statistics of the external large-scale random force $\mathbf{f}^v$.

In the approach based on the assumption of maximal randomness [10] the random forces are chosen to be Gaussian [11–13,17,18],

$$\langle f_j^v(x)\rangle=0, \quad \langle f_j^v(x_1)f_s^v(x_2)\rangle=D_{js}(x_1-x_2;g_{v_0}).$$

We choose the distribution function kernel to have the form [11]

$$D_{js}(x;g_{v_0})=g_{v_0}\,\delta(t)\int\frac{d^d\mathbf{k}}{(2\pi)^d}k^{4-d-2\epsilon}P_{js}(\mathbf{k})\,e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{2.3}$$

where $P_{js}(\mathbf{k})=\delta_{js}-k_jk_s/k^2$ is the transverse projection operator in the wave-vector space. For positive values of the coupling constant g_{v_0} Eq. (2.3) yields a positive-definite $(d\times d)$-square matrix of the forcing correlation functions. From Eq. (2.3) we see that temporal correlations of f^v have the character of white noise, while the spatial falloff of the correlations is controlled by the parameter ϵ. The matrix (2.3) is translation invariant and for the value $\epsilon=2$, becomes scale invariant. The value $\epsilon=2$ is physically most acceptable, since it represents the assumption that random forces act at very large scales, which substitutes for the effect of boundary conditions. For simplicity, we use the force correlation function (2.3) without the usual infrared regularization. The justification of this choice as well as the discussion of the central problem of the ϵ expansion, i.e., the continuation from $\epsilon=0$ to $\epsilon=2$, have been thoroughly discussed in Ref. [21].

We are working in an arbitrary dimension, but the renormalization will be carried out in relation to the two-dimensional model. Therefore, a few words on the choice of the forcing spectrum are in order. In two-dimensional turbulence two scaling regimes may occur corresponding to the direct enstrophy and the inverse energy cascades. Different forcing spectra are required to model these scaling regimes. If the random forcing used to maintain the stationary state is peaked at some finite wave number k_I, then the enstrophy pumping rate β (per unit mass) and the energy pumping rate ε are related by $\beta=k_I^2\varepsilon$. Thus, in the case of a finite energy pumping rate concentrated at very large spatial scales, the entrophy pumping rate vanishes and no direct enstrophy cascade can be maintained. There is no inviscid conservation law for enstrophy above two dimensions, and thus no enstrophy cascade either. In the double expansion scheme the main goal is, however, three-dimensional turbulence. Therefore, we use a forcing spectrum, which gives rise to the energy cascade. In the latter the energy spectrum obeys the Kolmogorov $-5/3$ law both in two and three dimensions.

As in critical dynamics (see, e.g., [16]), the stochastic problem (2.1), (2.2), and (2.3) is mapped to a quantum-field model, which is determined by an effective De Dominicis-Janssen ''action'' $S\{\mathbf{v},\mathbf{v}'\}$ constructed on the basis of the original stochastic problem. This action is a functional of the stochastic velocity field $\mathbf{v}$ and an independent transverse auxiliary field $\mathbf{v}'$.

In this approach, the generating functional $\mathcal{G}$ of the velocity correlation and response functions is the functional integral

$$\mathcal{G}(\mathbf{A}^v,\mathbf{A}^{v'})=\int\mathcal{D}\mathbf{v}\,\mathcal{D}\mathbf{v}'\exp\left[S\{\mathbf{v},\mathbf{v}\}+\int dx(\mathbf{A}^v\cdot\mathbf{v}+\mathbf{A}^{v'}\cdot\mathbf{v}')\right],$$

with the effective action

$$S\{\mathbf{v},\mathbf{v}'\}=\frac{1}{2}\int dx_1\times\int dx_2[\,\nu_0^3v_j'(x_1)D_{js}(x_1-x_2;g_{v_0})v_s'(x_2)\,]-\int dx\,\mathbf{v}'\cdot\boldsymbol{\mathcal{N}}(\{\mathbf{v}\};\nu_0), \tag{2.4}$$

where $\mathbf{A}^v$, $\mathbf{A}^{v'}$ are the source fields, which are equivalent to regular external forces. Here, and henceforth, sums over repeated indices are implicitly assumed. Thus, the stochastic problem is transferred to the calculation of functional integrals. They can be calculated perturbatively by means of the Feynman diagrammatic technique, which has been first applied to hydrodynamics by Wyld [10].

The statistical model of the advection of the passive scalar characterized by the concentration $c(x)$ in the turbulent environment (see, e.g., [14], [22]) is given by the system of equations,

$$\boldsymbol{\mathcal{N}}(\{\mathbf{v}\};\nu_0)=\mathbf{f}^v,$$
$$\partial_t c+(\mathbf{v}\cdot\boldsymbol{\nabla})c-\nu_0u_0\nabla^2c=f^c, \tag{2.5}$$

where u_0 is the inverse Prandtl number. The random source field f^c is assumed to be Gaussian and its exact distribution function will be specified later.

We have treated the scalar stochastic model (2.5) in the framework of the RG double-expansion scheme, which will be specified in the following.

In the application of the field-theoretic renormalization group to the large-scale behavior of statistical problems, formal ultraviolet divergences arise in the wave-vector space. It is customary [23] to classify and analyze these divergences in terms of *one-particle irreducible* (1PI) Green functions. These functions also provide a useful tool for the study of RG problems in the theory of the turbulence [12]. For our purposes we introduce two types of 1PI Green functions Γ that correspond schematically to connected Feynman diagrams with two or three external legs.

A detailed analysis [14] of the renormalization of the passive scalar model has shown that there are superficial divergences in the graphs corresponding to the 1PI Green functions $\Gamma^{\nu\nu'}$ and $\Gamma^{cc'}$ in the renormalization scheme of Refs. [11,12] applicable for space dimensions $d>2$. We will refer to the approach of [11,12] as the standard scheme. The 1PI

Green functions $\Gamma^{vvv'}$ and $\Gamma^{vcc'}$, which could, by standard power counting, give rise to the renormalization of the nonlinear terms in the Navier-Stokes and advection-diffusion equation, are actually finite due to the Galilei invariance of the stochastic equations with temporal white noise.

In two dimensions additional divergences in the graphs of the 1PI Green function $\Gamma^{v'v'}$ occur. Based on this, Ronis [24] has proposed a double-expansion approach using, in addition to the parameter ϵ in the force correlation function (2.3), the deviation δ of the space dimension from two as a small expansion parameter where $2\delta=d-2$. The main (false) conclusion of [24] was that the energy dissipation operator $(\nu_0/2)\partial_i v_j \partial_i v_j$ has a nonvanishing anomalous scaling dimension, i.e., it does not scale as the second power of $\partial_i v_j$.

The work of Ronis [24] was criticized by Honkonen and Nalimov [25]. The essence of their argument is that the nonlocal term

$$\int d^d\mathbf{x}_1 \int d^d\mathbf{x}_2\, \mathbf{v}'(\mathbf{x}_1,t)\cdot\mathbf{v}'(\mathbf{x}_2,t) \times \int \frac{d^d\mathbf{k}}{(2\pi)^d} k^{2-2\delta-2\epsilon} e^{i\mathbf{k}\cdot(\mathbf{x}_1-\mathbf{x}_2)}$$

generated by the force correlation function (2.3) is not renormalized since the divergences produced by the graphs are always local in space and time.

Renormalization is carried out at the critical values of the parameters $d=2$, $\epsilon=0$. The long-range part of the correlation function of the random force is a powerlike function of the wave number $\propto k^{2-2\delta-2\epsilon}$, and thus a singular function of the wave number at the origin. The renormalization gives rise to regular in the wave-number terms only; therefore, singular terms are not renormalized. When $\delta=\epsilon=0$, however, the correlation function becomes a regular function of the wave number and cannot be distinguished from the local counterterms $\propto k^2$. It is not obvious how the model should be renormalized in this case. The prescription proposed in Ref. [25] is based on the observation that, in order to distinguish between the original correlation function $\propto k^{2-2\delta-2\epsilon}$ and the local counterterms $\propto k^2$, an analytic regularization with, for instance, the parameter $\delta+\epsilon$ must be used. In order to make the model multiplicatively renormalizable the local term $\propto k^2$ is added to the force correlation function at the outset. Only this term is then renormalized, whereas the nonlocal term is left intact, contrary to the earlier treatment of the d-dimensional model near two dimensions [24].

Thus, a *local* term $\mathbf{v}'\cdot\nabla^2\mathbf{v}'$ must be added to the force correlation function for a consistent renormalization of the model. This leads to significant changes in the RG analysis. In particular, the anomalous scaling dimension of the energy dissipation operator turns out to vanish, as in the ϵ expansion for $d>2$ [12,26]. This conclusion is based on the analysis—in two dimensions—of the renormalization of all Galilei invariant scalar local operators of canonical dimension four (which is the canonical dimension of the energy dissipation operator) with zero wave number and frequency. All such operators may mix with each other under the renormalization [23]. This gives rise to a matrix of renormalization constants in the case of multiplicative renormalization of these operators. The analysis of the graphs carried out in Ref. [25] shows that the diagonal element corresponding to the energy dissipation operator in the matrix of renormalization constants is equal to unity. Due to the partially triangular structure of the matrix this implies that the anomalous dimension of the energy dissipation operator vanishes.

Inspired by the study of [25], we present here a generalized treatment of the model (2.5) near two dimensions, in which the force correlation function has both the long-range and local term at the outset.

The long-range term of the correlation function of the source field f^c gives rise to the following nonlocal term of the action,

$$\int d^d\mathbf{x}_1 \int d^d\mathbf{x}_2\, c'(\mathbf{x}_1,t)\, c'(\mathbf{x}_2,t) \times \int \frac{d^d\mathbf{k}}{(2\pi)^d} k^{2-2\delta-2a\epsilon} e^{i\mathbf{k}\cdot(\mathbf{x}_1-\mathbf{x}_2)}.$$

Here a free parameter a is used to control the power form of the passive scalar injection spectrum. For $\delta=\epsilon=0$ this correlation function generates logarithmic divergences in the graphs of the 1PI Green function $\Gamma^{c'c'}$. To take these into account in the multiplicative renormalization of the model, we introduce the corresponding local term $c'\nabla^2 c'$ in the scalar part of the effective action.

Let us consider the problem (2.5) with the forcing and source statistics,

$$\langle f_j^v(x_1) f^c(x_2)\rangle=0,$$

$$\langle f_j^v(x_1) f_s^v(x_2)\rangle=\nu_0^3 D_{js}(x_1-x_2;1,g_{v10},g_{v20}), \quad (2.6)$$

$$\langle f^c(x_1) f^c(x_2)\rangle=\frac{u_0^3\,\nu_0^3}{d-1} D_{jj}(x_1-x_2;a,g_{c10},g_{c20}),$$

where the correlation matrix D is defined by the relation,

$$D_{js}(x;A,B,C)=\delta(t_1-t_2)\int \frac{d^d\mathbf{k}}{(2\pi)^d} P_{js}(\mathbf{k})\, e^{i\mathbf{k}\cdot\mathbf{x}} \times[B\,k^{2-2\delta-2A\epsilon}+C\,k^2]. \quad (2.7)$$

The correlation function (2.7) reflects the detailed intrinsic statistical definition of forcing, whose consequences are thoroughly discussed in Ref. [27]. The necessity to introduce a combined forcing, and also to include the additional couplings g_{v20}, g_{c20} in order to obtain a multiplicatively renormalizable field theoretic model, is absent in the earlier formulation of this problem.

In analogy with Eq. (2.4) the stochastic problem of the passive scalar can be described by the field-theoretic action,

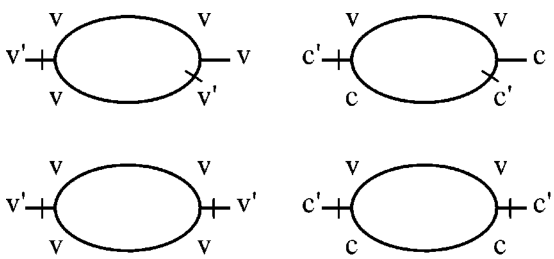

FIG. 1. One-loop graphs giving rise to the divergent terms in the perturbative expansion of the one-particle irreducible Green functions $\Gamma^{vv'}$, $\Gamma^{cc'}$, $\Gamma^{v'v'}$, and $\Gamma^{c'c'}$, respectively. The slashes on the lines denote the derivatives appearing in the cubic interaction terms of the action (2.8).

$$
\begin{aligned}
S^{PS}=&\frac{1}{2}\int dx_1 \int dx_2 \\
&\times\Bigg[\nu_0^3 v_j'(x_1)\, D_{js}(x_1-x_2;1,g_{v10},g_{v20}) v_s'(x_2) \\
&+\frac{u_0^3\,\nu_0^3}{d-1} c'(x_1)\, D_{jj}(x_1-x_2;a,g_{c10},g_{c20})\, c'(x_2)\Bigg] \\
&+\int dx\,\{c'[-\partial_t c+u_0\nu_0\nabla^2 c-(\mathbf{v}\cdot\boldsymbol{\nabla})c\,] \\
&-\mathbf{v}'\cdot\boldsymbol{\mathcal{N}}(\{\mathbf{v}\};\nu_0)\}. \qquad (2.8)
\end{aligned}
$$

As explained above, this action, containing nonanalytic terms (proportional to the coupling constants g_{v10} and g_{c10}), also requires analytic terms (proportional to g_{v20} and g_{c20}) in order to be multiplicatively renormalizable. All dimensional constants g_{v10}, g_{v20}, g_{c10}, and g_{c20}, which control the amount of randomly injected energy and mass given by Eqs. (2.6) and (2.7), play the role of the expansion parameters of the perturbation theory.

For convenience of further calculations the factors ν_0^3 and $\nu_0^3 u_0^3$ including the ''bare'' (molecular) viscosity ν_0 and the ''bare'' (molecular or microscopic) diffusion coefficient $\nu_0 u_0$ have been extracted.

III. CALCULATION OF THE FIXED POINTS OF THE RENORMALIZATION GROUP

Let us summarize the main points of the RG procedure which we have used. The general properties of the methods applied and perturbative techniques may be found, e.g., in [23].

The model (2.8) is renormalizable by the standard power-counting rules for $\delta=0$ and $\epsilon=0$. The divergent 1PI Green functions are $\Gamma^{vv'}$, $\Gamma^{cc'}$ as in the standard case [12] as well as $\Gamma^{v'v'}$, $\Gamma^{c'c'}$, which are typical of $d=2$. A graphical representation of the one-loop contributions to these functions is depicted in Fig. 1.

The lines in the graphs correspond to the following expressions as functions of time and wave vector:

$$
\Delta_{js}^{vv'}(\mathbf{k},t)=\Delta_{js}^{v'v}(-\mathbf{k},-t)=\theta(t)\,P_{js}(\mathbf{k})\,e^{-\nu_0 k^2 t}, \qquad (3.1)
$$

$$
\Delta^{cc'}(\mathbf{k},t)=\Delta^{c'c}(-\mathbf{k},-t)=\theta(t)\,e^{-u_0\nu_0 k^2 t},
$$

$$
\Delta_{js}^{vv}(\mathbf{k},t)=\tfrac{1}{2}\,\nu_0^2 P_{js}(\mathbf{k})(g_{v10}k^{-2\epsilon-2\delta}+g_{v20})e^{-\nu_0 k^2|t|},
$$

$$
\Delta^{cc}(\mathbf{k},t)=\tfrac{1}{2}\,u_0^2\,\nu_0^2(g_{c10}k^{-2a\epsilon-2\delta}+g_{c20})e^{-u_0\nu_0 k^2|t|}.
$$

The superscripts in Eq. (3.1) correspond to the notation of the lines of diagrams, and $\theta(t)$ is the step function.

We have used a combination of dimensional and analytic regularization [23]. The divergences appear in the form of poles in δ, ϵ and their linear combinations (which implies that δ and ϵ are treated as small parameters of same order). It is customary to use the minimal subtraction scheme in this connection [23]. In this approach we need to calculate only the divergent parts of the diagrams, since their finite parts are not needed in the one-loop approximation. Bearing this in mind, we (i) integrated over the internal time of the diagrams, (ii) integrated over the internal wave vectors of the diagrams with noninteger dimensionality $2+2\delta$ (dimensional regularization), and (iii) extracted the poles in δ, ϵ, $2\epsilon+\delta$, and $(a+1)\epsilon+\delta$ (for $\delta\to 0,\epsilon\to 0$), in agreement with the general minimal subtraction scheme [28].

The UV divergences can be removed by adding suitable counterterms to the basic action S_B obtained from Eq. (2.8) by the substitution of the renormalized parameters for the bare ones: $g_{v10}\to\mu^{2\epsilon}g_{v1}$, $g_{v20}\to\mu^{-2\delta}g_{v2}$, $g_{c10}\to\mu^{2a\epsilon+2\delta}g_{c1}$, $g_{c20}\to g_{c2}$, $\nu_0\to\nu$, and $u_0\to u$, where μ is a scale setting parameter having the same canonical dimension as the wave number.

The structure of the divergences together with the original form of the action (2.8) lead to the following counterterms:

$$
\begin{aligned}
\Delta S=\int dx\,[\,&\nu(1-Z_1)\,\mathbf{v}'\cdot\nabla^2\mathbf{v}+u\,\nu(1-Z_2)\,c'\nabla^2 c \\
&+\tfrac{1}{2}(Z_3-1)\nu^3 g_{v2}\mu^{-2\delta}\mathbf{v}'\nabla^2\mathbf{v}' \\
&+\tfrac{1}{2}(Z_4-1)u^3\nu^3 g_{c2}c'\nabla^2 c'\,].
\end{aligned}
$$

Within the combined dimensional and analytic renormalization which we used, the divergences appear in the form of a Laurent series in δ, ϵ, $2\epsilon+\delta$, and $(a+1)\epsilon+\delta$ and are absorbed in the renormalization constants Z_1, Z_2, Z_3, and Z_4.

A property essential for the application of the RG method is that the counterterms can be chosen in a form containing a finite number of terms of the same algebraic structure as the terms of the action (2.8). Then all UV divergences of the diagrams may be eliminated by a redefinition of the parameters of the original theory. After this, it is possible to continue the regularizing parameters to their ''physical'' values $\delta\to 1/2$, $\epsilon\to 2$, of which the former corresponds to the limit $(d\to 3)$, and the latter corresponds to the choice of the energy pumping rate as the only dimensional parameter of the forcing statistics.

Renormalized Green functions are expressed in terms of the renormalized parameters,

$$
g_{v1}=g_{v10}\,\mu^{-2\epsilon}Z_1^3, \quad g_{v2}=g_{v20}\,\mu^{2\delta}Z_1^3 Z_3^{-1},
$$

$$
g_{c1}=g_{c10}\,\mu^{-2a\epsilon-2\delta}Z_2^3, \quad g_{c2}=g_{c20}\,Z_2^3 Z_4^{-1}, \qquad (3.2)
$$

$$\nu=\nu_0 Z_1^{-1}, \quad u=u_0 Z_1 Z_2^{-1},$$

appearing in the renormalized action connected with the original action (2.8) by the relation of multiplicative renormalization,

$$S_R^{PS}\{\mathbf{v},\mathbf{v}',c,c',e\}=S^{PS}\{\mathbf{v},\mathbf{v}',c,c',e_0\},$$

where e is a shorthand for all the renormalized parameters $\{g_{v1},g_{v2},g_{c1},g_{c2},u,\nu\}$. Calculation of the correlation and response functions of the velocity and concentration fields with the use of the renormalized action yields renormalized Green functions without UV divergences.

The RG is mainly concerned with the prediction of the asymptotic behavior of the correlation and response functions expressed in terms of the anomalous dimensions γ_j by the use of β functions, defined via differential relations

$$\gamma_j=\mu\frac{\partial \ln Z_j}{\partial\mu}\bigg|_0, \quad \beta_g=\mu\frac{\partial g}{\partial\mu}\bigg|_0, \tag{3.3}$$

where $g=\{g_{v1},g_{v2},g_{c1},g_{c2},u\}$, and the subscript ''0'' refers to partial derivatives taken at fixed values of the bare (unrenormalized) parameters e_0.

All the UV divergences are present in the one-particle irreducible Green functions $\Gamma^{vv'}$, $\Gamma^{cc'}$, $\Gamma^{v'v'}$, and $\Gamma^{c'c'}$. The renormalization constants Z_1, Z_2, Z_3, and Z_4 can be extracted from the one-loop diagrams of Fig. 1.

The calculation yields in the minimal subtraction scheme,

$$Z_1=1-\frac{1}{64\pi}\left[\frac{g_{v1}}{\epsilon}-\frac{g_{v2}}{\delta}\right],$$

$$Z_2=1-\frac{1}{16\pi u(1+u)}\left[\frac{g_{v1}}{\epsilon}-\frac{g_{v2}}{\delta}\right],$$

$$Z_3=1-\frac{1}{64\pi}\left[\frac{g_{v1}^2}{g_{v2}}\frac{1}{2\epsilon+\delta}+\frac{2g_{v1}}{\epsilon}-\frac{g_{v2}}{\delta}\right], \tag{3.4}$$

$$Z_4=1-\frac{1}{16\pi u(1+u)}\left[\frac{g_{v1}g_{c1}}{g_{c2}}\frac{1}{(1+a)\epsilon+\delta}+\frac{g_{v1}}{\epsilon}+\frac{g_{v2}g_{c1}}{g_{c2}}\frac{1}{a\epsilon}-\frac{g_{v2}}{\delta}\right].$$

From the definitions (3.2) and (3.3) it follows that the β functions are

$$\beta_{v1}=g_{v1}(-2\epsilon+3\gamma_1), \quad \beta_{v2}=g_{v2}(2\delta+3\gamma_1-\gamma_3),$$

$$\beta_u=u(\gamma_1-\gamma_2), \tag{3.5}$$

$$\beta_{c1}=g_{c1}(-2\,a\,\epsilon-2\,\delta+3\gamma_2), \quad \beta_{c2}=g_{c2}(3\gamma_2-\gamma_4).$$

Here, the γ functions calculated from Eq. (3.4) are

$$\gamma_1=\frac{g_v}{32\pi}, \quad \gamma_2=\frac{g_v}{8\pi u(u+1)},$$

$$\gamma_3=\frac{g_v^2}{32\pi g_{v2}}, \quad \gamma_4=\frac{g_v(g_{c1}+g_{c2})}{8\pi g_{c2}u(u+1)},$$

where $g_v=g_{v1}+g_{v2}$.

Correlation functions of the fields are expressed in terms of scaling functions of the variable $s=k/\mu$, $s\in[0,1]$. The scale-invariant asymptotic behavior of the correlation functions stems from the existence of a stable fixed point of the RG. The continuous RG transformation is an operation linking a sequence of invariant parameters $\overline{g}(s)$ determined by the Gell-Mann Low equation,

$$\frac{d\overline{g}(s)}{d\ln s}=\beta_g[\overline{g}(s)], \tag{3.6}$$

with the abbrevation $\overline{g}=\{\overline{g}_{v1},\overline{g}_{v2},\overline{g}_{c1},\overline{g}_{c2},\overline{u}\}$, where the scaling variable s parametrizes the RG flow together with the initial conditions $\overline{g}|_{s=1}=g$. (In the infrared limit $s\to 0$.) Scale-invariant large-scale asymptotic behavior results at the infrared stable fixed point of Eqs. (3.6), determined by the system of equations $\beta_g(g^*)=0$, and the conditions $\overline{g}\to g^*$, when $s\to 0$.

In the vicinity of the fixed point all the trajectories $g(s)$ approach the fixed point, if the matrix $\Omega=(\partial\beta_g/\partial g)|_{g*}$ is positive definite. For $\overline{g}(s)$ close to g^* we obtain a system of linearized equations,

$$\left(I\,s\frac{d}{ds}-\Omega\right)(\overline{g}-g^*)=0,$$

where I is the 5×5 unit matrix. Solutions of this system behave like $\overline{g}=g^*+O(s^{\lambda_j})$, when $s\to 0$. The exponents λ_j, $j=1,2,3,4,5$ are the eigenvalues of the matrix Ω. The positive definiteness of Ω represented by the conditions $\mathrm{Re}(\lambda_j)\geqslant 0$ is the test of the infrared stability of the fixed point.

Due to the linearity of the advection-diffusion equation and Gaussian distribution of the source field f^c the connected correlation function W^{cc} of the concentration is a linear functional of the correlation function of the source field. This can be formally seen by taking the Gaussian integral over the fields c and c' in the generating functional,

$$\begin{aligned}\mathcal{G}(A^c,A^{c'})=&\int \mathcal{D}\mathbf{v}\,\mathcal{D}\mathbf{v}'\,\mathcal{D}c\,\mathcal{D}c'\\ &\times\exp\left[S^{PS}+\int dx\,(A^c\cdot c+A^{c'}\cdot c')\right]\\ =&\int \mathcal{D}\mathbf{v}\,\mathcal{D}\mathbf{v}'\exp\Bigg[S\{\mathbf{v},\mathbf{v}\}\\ &+\int dx_1\int dx_2\,A^c(x_1)\,L_v^{-1}(x_1-x_2)A^{c'}(x_2)\\ &+\int dx_1\int dx_2\,A^c(x_1)\\ &\times\frac{u_0^3\,\nu_0^3}{2(d-1)}L_v^{-1}D_{jj}L_v^{-T}(x_1-x_2)A^c(x_2)\Bigg],\end{aligned} \tag{3.7}$$

where L_v is the differential operator of the advection-diffusion equation,

$$L_v c = \partial_t c + (\mathbf{v}\cdot\boldsymbol{\nabla})c - \nu_0 u_0 \nabla^2 c.$$

Since the connected correlation function W^{cc} is the functional derivative

$$W^{cc}(x_1,x_2) = \left.\frac{\delta^2 \ln \mathcal{G}(A^c,A^{c'})}{\delta A^c(x_1)\,\delta A^c(x_2)}\right|_{A^c=A^{c'}=0},$$

it is obvious from the explicit expression (3.7) that W^{cc} is a linear functional of the correlation function $\langle f^c(x_1) f^c(x_2)\rangle = [u_0^3\nu_0^3/(d-1)]\,D_{jj}(x_1-x_2;a,g_{c10},g_{c20})$. In the perturbation expansion this means that each graph of the 1PI Green function $\Gamma^{c'c'}$ contains exactly one line Δ^{cc}, whereas $\Gamma^{cc'}$ does not contain Δ^{cc} at all. Therefore, the renormalization constant Z_2 is independent of the coupling constants g_{c1} and g_{c2}, and the dependence of Z_4 on g_{c1} and g_{c2} has the following simple form:

$$Z_4 = 1 - \zeta_2(g_{v1},g_{v2},u) - \frac{g_{c1}}{g_{c2}}\zeta_1(g_{v1},g_{v2},u).$$

Correspondingly, we may write for the function γ_4 the expression

$$\gamma_4 = \gamma_4'(g_{v1},g_{v2},u) + \frac{g_{c1}}{g_{c2}}\gamma_4''(g_{v1},g_{v2},u).$$

As a consequence, the RG functions β_{c1} and β_{c2} are linear functions of the coupling constants g_{c1} and g_{c2},

$$\beta_{c1} = g_{c1}(-2a\epsilon - 2\delta + 3\gamma_2),$$

$$\beta_{c2} = g_{c2}(3\gamma_2 - \gamma_4') - g_{c1}\gamma_4''. \tag{3.8}$$

Since this is an exact relation, it means that the parameters g_{c1} and g_{c2} of the source field correlation function play a role similar to that of the viscosity ν: They do not have any finite fixed-point values themselves, but their asymptotic behavior is governed by the fixed point of the RG (which is determined by the β functions corresponding to the renormalized parameters g_{v1}, g_{v2} of the force correlation function and the renormalized inverse Prandtl number u). Since the functions γ_1, γ_2, and γ_3 are functions of g_{v1}, g_{v2} and u only, vanishing of the first three β functions in Eq. (3.5) already yields a closed system of equations for the fixed point values of g_{v1}, g_{v2} and u. When approaching such a fixed point, the coefficient functions γ_2, γ_4', and γ_4'' in Eq. (3.8) approach their fixed point values, which determine the scaling dimensions of the coupling constants g_{c1} and g_{c2} at the fixed point.

Thus, in the vicinity of the fixed point determined by the system of equations,

$$g_{v1}(-2\epsilon + 3\gamma_1) = 0,$$

$$g_{v2}(2\delta + 3\gamma_1 - \gamma_3) = 0, \tag{3.9}$$

$$u(\gamma_1 - \gamma_2) = 0,$$

the running coupling constants of the correlation function of the source field have the following scaling behavior:

$$\bar{g}_{c1} = g_{c1}\left(\frac{k}{\mu}\right)^{\Delta_1},$$

$$\bar{g}_{c2} = g_{c2}\left(\frac{k}{\mu}\right)^{\Delta_2} + \frac{\gamma_4''^*}{2a\epsilon + 2\delta - \gamma_4'^*}\,g_{c1}\left[\left(\frac{k}{\mu}\right)^{\Delta_1} - 1\right], \tag{3.10}$$

where the scaling dimensions are

$$\Delta_1 = -2a\epsilon - 2\delta + 3\gamma_2^*, \quad \Delta_2 = 3\gamma_2^* - \gamma_4'^*,$$

and the asterisks denote the values of the γ functions at the fixed point (3.9).

Apart from the Gaussian fixed point $g_{v1}^* = g_{v2}^* = 0$, which is stable for $\delta > 0$, $\epsilon < 0$, there are two nontrivial fixed points of the RG: the fixed point corresponding to short-range correlations of the random force [13] with

$$g_{v1}^* = 0, \quad g_{v2}^* = -32\pi\delta,$$

and the inverse Prandtl number

$$u^* = \frac{\sqrt{17}-1}{2} \simeq 1.562. \tag{3.11}$$

The region of stability of this short-range fixed point $2\delta + 3\epsilon < 0$, $\delta < 0$ is determined by the positivity of the eigenvalues of the Ω matrix

$$\lambda_1 = -2\epsilon - 3\delta, \quad \lambda_2 = -2\delta, \quad \lambda_3 = -2\delta\frac{\sqrt{17}}{\sqrt{17}+1}.$$

The third fixed point is the *kinetic* fixed point, which is the fixed point relevant to the description of turbulent diffusion. At the kinetic fixed point the value of the renormalized inverse Prandtl number u is given by Eq. (3.11) and the values of the other relevant coupling constants

$$g_{v1}^* = \frac{64\pi}{9}\frac{\epsilon(2\epsilon + 3\delta)}{\epsilon + \delta}, \quad g_{v2}^* = \frac{64\pi}{9}\frac{\epsilon^2}{\delta + \epsilon}, \tag{3.12}$$

which confirm those obtained for the stochastic Navier-Stokes equation earlier [25].

The calculation of the Ω matrix [up to the order $O(\epsilon,\delta)$] at this fixed point yields the eigenvalues,

$$\lambda_{1,2} = \tfrac{1}{3}[(4\delta + 3\epsilon) \pm \sqrt{9\epsilon^2 - 12\delta\epsilon - 8\delta^2}],$$

$$\lambda_3 = \frac{4\sqrt{17}}{\sqrt{17}+1}\epsilon.$$

From these expressions we see that the region of stability of the kinetic fixed point is $\epsilon > 0$, $2\epsilon + 3\delta > 0$.

IV. KOLMOGOROV AND OBUKHOV-CORRSIN CONSTANTS

We use the notation,

$$W_{ij}^{vv}(g,\mathbf{k})=\int\frac{d^d\mathbf{x}_1}{(2\pi)^d}\langle v_i(\mathbf{x}_1,t)\,v_j(\mathbf{x}_2,t)\rangle\,e^{i\mathbf{k}\cdot(\mathbf{x}_1-\mathbf{x}_2)},$$

$$W^{cc}(g,k)=\int\frac{d^d\mathbf{x}_1}{(2\pi)^d}\langle c(\mathbf{x}_1,t)\,c(\mathbf{x}_2,t)\rangle\,e^{i\mathbf{k}\cdot(\mathbf{x}_1-\mathbf{x}_2)},$$

for the Fourier transforms of the equal-time pair correlation functions of the random velocity field and the passive scalar, respectively. To calculate the Kolmogorov and Obukhov-Corrsin (or Batchelor) constants we construct the large-scale asymptotic expressions for the correlation functions.

The unrenormalized correlation functions are independent of the scale-setting parameter μ of the renormalized correlation functions. This property leads to the basic renormalization group equation for stationary W^{vv},

$$\mu\partial_\mu|_{e_0}W^{vv}(g,\mathbf{k})=[\,\mu\partial_\mu+\beta_g\partial_g-\gamma_1\,\nu\,\partial_\nu]\,W^{vv}(g,\mathbf{k})=0, \tag{4.1}$$

with W^{cc} satisfying the same equation exactly. Here, $\beta_g\partial_g$ is a shorthand for

$$\beta_g\partial_g=\beta_{gv1}\partial_{gv1}+\beta_{gv2}\partial_{gv2}+\beta_{gc1}\partial_{gc1}+\beta_{gc2}\partial_{gc2}+\beta_u\partial_u\,.$$

It is convenient to single out the dimensionless scaling functions R_v and R_c defined by the relations,

$$W_{ij}^{vv}(g,k)=\nu^2\,k^{-2\delta}s^{-2\epsilon}P_{ij}R_v(g,s),$$

$$W^{cc}(g,\mathbf{k})=\nu^2\,s^{-2\delta-2a\epsilon}R_c(g,s), \tag{4.2}$$

where the dimensionless wave number $s=k/\mu$ has been introduced. From Eqs. (4.1) and (4.2) the basic renormalization group equations for the scaling functions are

$$(\,-s\partial_s+\beta_g\partial_g+2\epsilon-2\gamma_1)\,R_v(g,s)=0,$$

$$(-s\partial_s+\beta_g\partial_g+2\delta+2a\epsilon-2\gamma_1)\,R_c(g,s)=0. \tag{4.3}$$

Solving by the method of characteristics, we obtain for Eq. (4.3) the solution

$$R_v(g,s)=R_v(\overline{g},1)s^{2\epsilon}e^{-2\int_1^s dx\;\gamma_1(\overline{g}(x))/x},$$

$$R_c(g,s)=R_c(\overline{g},1)s^{2\delta+2a\epsilon}e^{-2\int_1^s dx\;\gamma_1(\overline{g}(x))/x}, \tag{4.4}$$

where $\overline{g}$ is the solution of the Gell-Mann Low Eqs. (3.6). The solution (4.4) leads to scale-invariant behavior of the correlation functions governed by the infrared stable fixed point.

To relate the general solution (4.4) to the physically relevant injection rates of energy and the passive scalar, we express the renormalized correlation functions in terms of the unrenormalized parameters, which, in turn, may be connected with the injection rates. Due to the connection between the functions β_{v1} and γ_1, the scaling factor in Eq. (4.4) may be calculated in the closed form [21],

$$\nu^2e^{-2\int_1^s dx\;\gamma_1(\overline{g}(x))/x}=\left(\frac{g_{v10}\nu_0^3}{\overline{g}_{v1}}\right)^{2/3}k^{-4\epsilon/3}. \tag{4.5}$$

Similarly, the combination $\overline{u}^2\overline{g}_{c1}$ appearing in the function $R_c(\overline{g},1)$ may be expressed as

$$\overline{u}^2\overline{g}_{c1}=\frac{\overline{g}_{v1}}{\overline{u}}\,\frac{g_{c10}u_0^3}{g_{v10}}k^{2(1-a)\epsilon-2\delta}. \tag{4.6}$$

In the asymptotic regime governed by the kinetic fixed point we thus obtain from Eqs. (4.2), (4.4), (4.5), and (4.6)—at leading order—the correlation functions

$$W_{ij}^{vv}(g,\mathbf{k})=\frac{1+2\delta}{2}(g_{v1}^*+g_{v2}^*)P_{ij}\left(\frac{g_{v10}\nu_0^3}{g_{v1}^*}\right)^{2/3}k^{-2\delta-4\epsilon/3},$$

$$W^{cc}(g,\mathbf{k})=\frac{(g_{v1}^*)^{1/3}}{2u^*}\,\frac{g_{c10}\nu_0^3u_0^3}{(g_{v10}\nu_0^3)^{1/3}}\times\left(\frac{2a\epsilon+2\delta}{2a\epsilon+2\delta-\gamma'^*_4}\right)k^{-2a\epsilon-2\delta+2\epsilon/3}, \tag{4.7}$$

for $a>1/12$, which is sufficient, since the physically interesting value is $a=1$. It should be noted that in the scaling function R_c the asymptotic behavior of the running coupling constants $\overline{g}_{c1}$ and $\overline{g}_{c2}$ is given by the expressions (3.10), whereas the coupling constants $\overline{g}_{v1}$, $\overline{g}_{v2}$ and $\overline{u}$ approach their fixed-point values. We have taken into account in Eq. (4.7) that only terms with the scaling dimension Δ_1 survive from Eq. (3.10).

The powers of the wave number in Eq. (4.7) are exact, whereas the coefficients are calculated at leading order of the perturbation expansion. At the kinetic fixed point (3.11) and (3.12), this yields the leading order term of the ϵ, δ expansion of the scaling functions R_v and R_c.

The energy injection rate ε and the scalar injection rate χ may be expressed as

$$\varepsilon=\frac{1}{2}\int\frac{d^d\mathbf{k}}{(2\pi)^d}\langle\mathbf{f}^v(\mathbf{k})\cdot\mathbf{f}^v(-\mathbf{k})\,\rangle,$$

$$\chi=\int\frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d}\langle f^c(\mathbf{k})f^c(-\mathbf{k})\rangle. \tag{4.8}$$

For our choice of correlation functions after the introduction of simple sharp cutoffs, Eq. (4.8) yields the relation between the unrenormalized values of the coupling constants and the energy and scalar injection rates in the form

$$\varepsilon=\frac{(d-1)\nu_0^3}{2}\int_{k_I<k<k_d}\frac{d^d\mathbf{k}}{(2\pi)^d}(g_{v10}k^{2-2\delta-2\epsilon}+g_{v20}k^2),$$

$$\chi=\nu_0^3\,u_0^3\int_{k_I<k<k_d'}\frac{d^d\mathbf{k}}{(2\pi)^d}(\,g_{c10}k^{2-2\delta-2a\epsilon}+g_{c20}k^2\,), \tag{4.9}$$

where k_I is the wave number corresponding to the integral scale, and k_d, k_d' are the characteristic wave numbers of viscous and molecular dissipation, respectively.

In the stationary state modeling developed isotropic turbulence, the energy and mass injection are assumed to take place at large scales. Therefore, we put the parameters g_{v20} and g_{c20}, which correspond to small-scale injection of energy and scalar, equal to zero. It should be borne in mind that the corresponding running coupling constants are created in the course of renormalization regardless of the unrenormalized values of these parameters. The present perturbative calculation yields only the leading order in the ϵ, δ expansion of the amplitude coefficients in the scaling form of the correlation functions. Therefore, the coupling constants should be solved from Eq. (4.9) as functions of ε and χ also only at leading order of ϵ, δ expansion. Thus, we arrive at the relations

$$\varepsilon=\frac{\nu_0^3 g_{v10}}{16\pi}k_d^{4-2\epsilon},$$

$$\chi=\frac{\nu_0^3 u_0^3 g_{c10}}{8\pi}k_d'^{\,4-2a\epsilon}, \tag{4.10}$$

valid for large Reynolds and Peclet numbers, when $k_d/k_I \sim \mathrm{Re}^{3/4}\gg 1$ and $k_d'/k_I\sim \mathrm{Pe}^{3/4}\gg 1$.

We use the following definitions for the d-dimensional and one-dimensional spectra of the energy and passive scalar,

$$W_{ii}^{vv}(t,\mathbf{x};t,\mathbf{x})=2\int_0^\infty dk\,E(k),$$

$$W_{11}^{vv}(t,\mathbf{x};t,\mathbf{x})=\int_0^\infty dk_1\,E_1(k_1), \tag{4.11}$$

$$W^{cc}(t,\mathbf{x};t,\mathbf{x})=\int_0^\infty dk\,E^c(k)=\int_0^\infty \mathrm{d}k_1\,E_1^c(k_1),$$

respectively. The spectral functions E, E_1 and E^c, E_1^c are connected as

$$E_1(k_1)=\frac{2\Gamma(d/2)}{\sqrt{\pi}\Gamma((d+1)/2)}\int_{k_1}^\infty\left(1-\frac{k_1^2}{k^2}\right)^{(d-1)/2}E(k)\frac{dk}{k},$$

$$E_1^c(k_1)=\frac{2\Gamma(d/2)}{\sqrt{\pi}\Gamma((d-1)/2)}\int_{k_1}^\infty\left(1-\frac{k_1^2}{k^2}\right)^{(d-3)/2}E^c(k)\frac{dk}{k}. \tag{4.12}$$

From Eqs. (4.7), (4.10), and (4.11) we then obtain the following asymptotic expressions for the spectra:

$$E(k)=C_k\,\varepsilon^{2/3}k^{1-4\epsilon/3}k_d^{4(\epsilon-2)/3},$$

$$E_1(k_1)=C_{k1}\,\varepsilon^{2/3}k_1^{1-4\epsilon/3}k_d^{4(\epsilon-2)/3},$$

$$E^c(k)=C_{oc}\,\chi\,\varepsilon^{-1/3}k^{1-2a\epsilon+2\epsilon/3}k_d^{(4-2\epsilon)/3}k_d'^{\,2a\epsilon-4}, \tag{4.13}$$

$$E_1^c(k_1)=C_{oc1}\,\chi\,\varepsilon^{-1/3}k_1^{1-2a\epsilon+2\epsilon/3}k_d^{(4-2\epsilon)/3}k_d'^{\,2a\epsilon-4}.$$

Here, the Kolmogorov constant in the d-dimensional spectrum is

$$C_k=(2\pi)^{-1/3}\frac{g_{v1}^*+g_{v2}^*}{(g_{v1}^*)^{2/3}}=\frac{2\times12^{1/3}\,\epsilon^{1/3}(\epsilon+\delta)^{2/3}}{(2\epsilon+3\delta)^{2/3}}, \tag{4.14}$$

and coincides for $\delta=0$ with the value obtained in [29]. It is, however, different from the value $C_k=2^{4/3}(2+\delta)^{1/3}\epsilon^{1/3}3^{-1/3}$ [21] obtained in the standard ϵ expansion.

The Obukhov-Corrsin constant in Eq. (4.13) for the d-dimensional spectrum is

$$C_{oc}=(2\pi)^{-1/3}\frac{2a\epsilon+2\delta}{2a\epsilon+2\delta-\gamma_4'^*}\,\frac{(g_{v1}^*)^{1/3}}{u^*}$$
$$=\frac{4\times12^{1/3}\,\epsilon^{1/3}(2\epsilon+3\delta)^{1/3}(a\epsilon+\delta)}{(\sqrt{17}-1)(\epsilon+\delta)^{1/3}(3a\epsilon+3\delta-\epsilon)}. \tag{4.15}$$

For one-dimensional spectrum the conversion factors are

$$\frac{C_{k1}}{C_k}=\frac{\Gamma(1+\delta)\Gamma(2\epsilon/3-1/2)}{\sqrt{\pi}\Gamma(1+\delta+2\epsilon/3)},$$

$$\frac{C_{oc1}}{C_{oc}}=\frac{\Gamma(1+\delta)\Gamma(a\epsilon-\epsilon/3-1/2)}{\sqrt{\pi}\Gamma(\delta+a\epsilon-\epsilon/3)}.$$

Contrary to the standard RG approach [30,31], the ratio C_k/C_{oc} here is not a function of the fixed-point value of the inverse Prandtl number only, but depends on the fixed-point values of other coupling constants as well. In this sense this ratio seems less universal than in the standard approach. Due to the linear dependence of the scalar-scalar correlation function on the correlation function of the random scalar source, however, the ratio C_k/C_{oc} is definitely independent of the parameters of the scalar injection. It is a curious feature of the expressions (4.14) and (4.15) that the ratio C_k/C_{oc} becomes independent of both δ and ϵ for $a=1$, which is the physical value of this parameter.

In the inertial-convective range [32] the only relevant physical parameters should be the energy and scalar injection rates. To get rid of the dissipative wave numbers, we choose $\epsilon=2$ and $a=1$ as the physical values of these parameters corresponding to the inertial-convective range, in which the Prandtl number u_0^{-1} and, consequently, the ratio k_d/k_d' are of the order of unity. Analysis of the limiting cases $u_0\gg1$ (inertial-conductive range) and $u_0\ll1$ (viscous-convective range) in the present setup requires more detailed knowledge about the scaling function R_c and is beyond the scope of the present treatment.

Thus, in the inertial-convective range, we arrive at the values

$$C_k=2^{4/3}\cdot3^{1/3}\approx3.634,\quad C_{k1}=\frac{\Gamma(5/6)}{\sqrt{\pi}\Gamma(7/3)}C_k\approx1.944,$$

$$d=2,$$

$$C_k = 4\,(75/121)^{1/3} \approx 3.411, \quad C_{k1} = \frac{18}{55} C_k \approx 1.116, \quad d=3 \tag{4.16}$$

for the Kolmogorov constant and

$$C_{oc} = \frac{2\times 2^{4/3}\cdot 3^{1/3}}{\sqrt{17}-1} \approx 2.327,$$

$$C_{oc1} = \frac{\Gamma(5/6)}{\sqrt{\pi}\,\Gamma(4/3)}$$

$$C_{oc} \approx 1.660, \quad d=2,$$

$$C_{oc} = \frac{8\,(75/121)^{1/3}}{\sqrt{17}-1} \approx 2.184,$$

$$C_{oc1} = \frac{3}{5} C_{oc} \approx 1.310, \quad d=3 \tag{4.17}$$

for the Obukhov-Corrsin constant. When comparing with the data from experiments and simulations, which yield the values $C_{k1}\approx 0.5$ [33] and $C_{oc1}\approx 0.4$ [34] $(d=3)$, it should be borne in mind that the values (4.16) and (4.17) correspond to the leading terms of an asymptotic expansion with an expansion parameter, which is not small. The value of the ratio $C_{oc1}/C_{k1} = 11/[3(\sqrt{17}-1)] \approx 1.174$, however, is a bit closer to the experimental value 0.8 [34] than the values of the constants themselves.

The value of the Kolmogorov constant C_k obtained here for $d=3$ is also different from that in the standard ϵ expansion, where $C_k(d=3) = 2(10/3)^{1/3} = 2.988$ [21]. Of course, the leading order terms of two different expansions of the Kolmogorov constant do not allow us to make definite conclusions about the eventual value of this constant in these expansions. On the other hand, the coefficients of the universal powers in scaling regimes predicted by the RG are not universal in general, and this may well be the case here as well.

In the minimal subtraction scheme only the singular contributions of the graphs to the renormalization constants are retained. In general, the renormalization constants are determined up to a finite renormalization, which may be used to relate the parameters of the model to observables at some reference scale. For the stochastic Navier-Stokes equation, for instance, a natural choice would be

$$\frac{1}{2}\frac{\partial^2}{\partial k^2}(W^{vv'})^{-1}(\omega,k)\Big|_{\substack{\omega=\mu^2\nu\\ k=\mu}} = -\nu_0, \tag{4.18}$$

where $W^{vv'}$ is the renormalized response function of the velocity field, and ν_0 is the viscosity at the reference wavenumber μ. The normalization (4.18) implies that at wave numbers of the order of μ the nonlinear terms are negligible.

The choice of renormalization scheme does not affect the scaling exponents, but it may change the scaling functions. However, in the ϵ expansion, any renormalization prescription different from the minimal subtraction scheme changes the coefficient and scaling functions by terms, which are of higher order than the leading order in the ϵ expansion. In the present paper the correlation functions and spectra are calculated at leading order in the ϵ expansion, which is uniquely given by the minimal subtraction procedure.

V. CONCLUSIONS

In the present paper we have investigated the model of diffusion of passive admixture in a turbulent velocity field generated by the stochastic Navier-Stokes equation. For a long-range correlated random force in the Navier-Stokes equation and a long-range correlated random injection of the passive scalar in the advection-diffusion equation, we have constructed the field-theoretic action corresponding to the stochastic problem.

In order to make the field-theoretic model multiplicatively renormalizable at two dimensions we have added short-range terms to the correlation functions of the random source fields and renormalized the model at one-loop order.

The principal consequences of the multiplicative renormalizability conditions are (i) a more complex form of the action [25], or equivalently (ii) a more complex forcing correlation function [27] including large and small scale terms (2.7).

By use of the renormalization group, we have constructed an expansion of the model in two dimensionless parameters: the deviation of the space from two and the deviation of the exponent of the powerlike correlation functions of the random source fields from the logarithmic value.

We have found that the renormalization constants of the model are independent of the coupling constants g_{c1} and g_{c2}, which measure the scalar injection rate, apart from the renormalization constant of the scalar injection rate term with local correlations, which is shown to be a linear function of these coupling constants to all orders in the perturbation theory. As a consequence, the large-scale asymptotic behavior of the model is determined by the fixed point of the model without scalar injection.

We have calculated the energy and scalar injection spectra at the leading order of the ϵ, δ expansion and found them to reproduce the Kolmogorov and Batchelor scaling behavior for the physically most acceptable values of the expansion parameters. The Kolmogorov and Obukhov-Corrsin (Batchelor) constants have also been calculated at the leading order of the double expansion. The ratio of the Kolmogorov and Obukhov-Corrsin constants depends not only on the fixed-point value of the inverse Prandtl number, as in the standard ϵ expansion, but also on the fixed-point values of the rate parameters of the energy injection. It is, however, independent of the parameters of the scalar injection.

The standard ϵ expansion [11,12] and the present double expansion are the only regular schemes to construct scaling functions (correlation functions, spectra etc.) describing the self-similar behavior in stochastically forced hydrodynamic equations. There is recent evidence that the results of the standard ϵ expansion are in agreement with nonperturbative results in the simple Kraichnan model for the advection of the passive scalar for finite values of ϵ [35]. This allows us to hope that the results in the ϵ expansion and the double expansion will also be useful for more realistic velocity fields for those finite values of the expansion parameters that correspond to the Kolmogorov scaling.

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